TAMENESS OF LOCAL COHOMOLOGY OF MONOMIAL IDEALS WITH RESPECT TO MONOMIAL PRIME IDEALS

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ABSTRACT. In this paper we consider the local cohomology of monomial ideals with respect to monomial prime ideals and show that all these local cohomology modules are tame.

INTRODUCTION

Let R be a graded ring. Recall that a graded R-module N is tame, if there exists an integer j_0 such that $N_j = 0$ for all $j \leq j_0$, or else $N_j \neq 0$ for all $j \leq j_0$. Brodmann and Hellus [4] raised the question whether for a finitely generated, positively graded algebra R with R_0 Noetherian, the local cohomology modules $H^i_{R_+}(M)$ for a finitely generated graded R-module M are all tame. Here $R_+ = \bigoplus_{i>0} R_i$ is graded irrelevant ideal of R. See [1] for a survey on this problem.

In this paper we only consider rings defined by monomial relations. We first consider the squarefree case, since from a combinatorial point of view this is the more interesting case and also since in this case the formula which we obtain are more simple. So let K be a field and Δ a simplicial complex on $V = \{v_1, \ldots, v_n\}$. In Section 1, as a generalization of Hochster's formula [3], we compute (Theorem 1.3) the Hilbert series of local cohomology of the Stanley-Reisner ring $K[\Delta]$ with respect to a monomial prime ideal. With the choice of the monomial prime ideal, the Stanley-Reisner ring $K[\Delta]$, and hence also all the local cohomology of it, can be given a natural bigraded structure. In Proposition 1.7 we give a formula for the K-dimension of the bigraded components of the local cohomology modules. Using this formula we deduce that the local cohomology of $K[\Delta]$ with respect to a monomial prime ideal is always tame.

In [6] Takayama generalized Hochster's formula to any graded monomial ideal which is not necessarily squarefree. In Section 2, as a generalization of Takayama's result, we compute the Hilbert series of local cohomology of monomial ideals with respect to monomial prime ideals and observe that again all these modules are tame. The result proved here is surprising because in a recent paper, Cutkosky and Herzog [5] gave an example which shows that in general not all local cohomology modules are tame.

1. LOCAL COHOMOLOGY OF STANLEY-REISNER RINGS WITH RESPECT TO MONOMIAL PRIME IDEALS

Let K be a field and let $S = K[Y_1, \ldots, Y_r]$ be a polynomial ring with the standard grading. For a squarefree monomial ideal $I \subset S$ we set $R = S/I$. We denote by

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y_i the residue classes of indeterminates Y_i in R for $i = 1, \ldots, n$. Thus we have $R = K[y_1, \ldots, y_r]$. We may view R as the Stanley-Reisner ring of some simplicial complex Δ with vertices $\{w_1, \ldots, w_r\}$.

Let P be any monomial prime ideal of R. We may assume that $P = (y_1, \ldots, y_n)$ for some integer $n \leq r$. After this choice of P we view R as a bigraded K-algebra. We rename some of the variables, and set $x_i = y_{n+i}$ for $i = 1, \ldots, m$ where $m = r-n$, and assign the following bidegrees: $\deg x_i = (1, 0)$ for $i = 1, \ldots, m$ and $\deg y_j = (0, 1)$ for $j = 1, \ldots, n$. We decompose the vertex set of the corresponding simplicial complex Δ accordingly, so that Δ has vertices $\{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ where vertices $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_n\}$ correspond to the variables of $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, respectively. By [1, Theorem 5.1.19] we have

$$H_P^i(R) \cong H^i(C^\bullet) \quad \text{for all} \quad i \geq 0,$$

where $C^\bullet$ is the Čech complex

$$C^\bullet : 0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

with

$$C^t = \bigoplus_{1 \leq j_1 < \cdots < j_t \leq n} R_{y_{j_1} \cdots y_{j_t}},$$

and whose differential is composed of the maps

$$(-1)^{s-1} nat : R_{y_{j_1} \cdots y_{j_t}} \longrightarrow R_{y_{j_1} \cdots y_{j_{t+1}}},$$

if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \hat{j}_s, \ldots, j_{t+1}\}$ and 0 otherwise. Note that $C^\bullet$ is a $\mathbb{Z}^m \times \mathbb{Z}^n$-bigraded complex. For $(a, b) \in \mathbb{Z}^m \times \mathbb{Z}^n$ and $y = y_{j_1} \ldots y_{j_s}$ with $1 \leq j_1 < \cdots < j_s \leq n$ one defines a $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading on R_y by setting

$$(1) \qquad\qquad (R_y)_{(a,b)} = \{r/y^l : \deg r - l \deg y = (a, b)\}.$$

Here r is a bihomogeneous element in R, $l \in \mathbb{Z}$ and deg denotes the multi-bidegree. Given $F = \{w_{j_1}, \ldots, w_{j_s}\} \subseteq W$ and $b \in \mathbb{Z}^n$. We set $G_b = \{w_j : w_1 \leq w_j \leq w_n, b_j < 0\}$, $H_b = \{w_j : w_1 \leq w_j \leq w_n, b_j > 0\}$ and the support of b is the set $\operatorname{supp} b = \{w_j : w_1 \leq w_j \leq w_n, b_j \neq 0\}$. Note that $\operatorname{supp} b = G_b \cup H_b$.

We set $N_a = \{v_i : v_1 \leq v_i \leq v_m, a_i \neq 0\} = \operatorname{supp} a$ for $a \in \mathbb{Z}^m$ and denote by $\mathbb{Z}_+^m$ and $\mathbb{Z}_-^n$ the sets of $\{a \in \mathbb{Z}^m : a_i \geq 0 \text{ for } i = 1, \ldots, m\}$ and $\{b \in \mathbb{Z}^n : b_i \leq 0 \text{ for } i = 1, \ldots, n\}$, respectively. With the notation introduced one has

Lemma 1.1. *The following statements hold:*
   (a) $\dim_K(R_y)_{(a,b)} \leq 1$, *for all* $a \in \mathbb{Z}^m$ *and* $b \in \mathbb{Z}^n$.
   (b) $(R_y)_{(a,b)} \cong K$, *if and only if* $F \supset G_b$, $F \cup H_b \cup N_a \in \Delta$ *and* $a \in \mathbb{Z}_+^m$.

Proof. As explained before, we may view the standard graded polynomial ring S as a standard bigraded polynomial ring and then $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ with $m + n = r$ is also naturally bigraded. Thus part (a) follows from [3, Lemma 5.3.6 (a)]. For the proof (b) we set $c = (a, b)$. By [3, Lemma 5.3.6 (b)] we have $F \supset G_c$ and $F \cup H_c \in \Delta$. Thus (1) implies that $a \in \mathbb{Z}_+^m$ and hence $G_c = G_b$. We also note that $H_c = H_b \cup N_a$. $\qquad\square$

As a consequence of Lemma 1.1 for $a \in \mathbb{Z}_+^m$, $b \in \mathbb{Z}^n$ and $i \in \mathbb{Z}$ we observe that $(C^i)_{(a,b)}$ has the following K- basis:

$$\{b_F : F \supset G_b, F \cup H_b \cup N_a \in \Delta, |F| = i\}.$$

Therefore, since $C^\bullet$ is $\mathbb{Z}^m \times \mathbb{Z}^n$-bigraded complex one obtains for each $(a, b) \in \mathbb{Z}^m \times \mathbb{Z}^n$ a complex

$$(C^\bullet)_{(a,b)} : 0 \to (C^0)_{(a,b)} \to (C^1)_{(a,b)} \to \cdots \to (C^n)_{(a,b)} \to 0,$$

of finite dimensional K-vector spaces

$$(C^i)_{(a,b)} = \bigoplus_{\substack{F \supset G_b \\ F \cup H_b \cup N_a \in \Delta \\ |F| = i}} K b_F.$$

The differential $\partial : (C^i)_{(a,b)} \longrightarrow (C^{i+1})_{(a,b)}$ is given by $\partial(b_F) = \sum (-1)^{\partial(F,F')} b_{F'}$ where the sum is taken over all F' such that $F' \supset F$, $F' \cup H_b \cup N_a \in \Delta$ and $|F'| = i+1$, and where $\partial(F, F') = s$ for $F' = [w_0, \ldots, w_i]$ and $F = [w_0, \ldots, \hat{w}_s, \ldots, w_i]$. Then we describe the (a, b)th component of the local cohomology in terms of this subcomplex:

$$(2) \qquad H_P^i(K[\Delta])_{(a,b)} = H^i(C^\bullet)_{(a,b)} = H^i(C_{(a,b)}^\bullet).$$

Let Δ be a simplicial complex with vertex set V and $\tilde{\mathcal{C}}(\Delta)$ the augumented oriented chain complex of Δ, see [3, Section 5.3] for details. For an abelian group G, the ith reduced simplicial cohomology of Δ with values in G is defined to be

$$(3) \qquad \widetilde{H}^i(\Delta; G) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}(\Delta), G)) \quad \text{for all} \quad i.$$

Given $F \subseteq V$, we recall the following definitions : The *star* of F is the set $st_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}$, and the *link* of F is the set $lk_\Delta F = \{G : F \cup G \in \Delta, F \cap G = \emptyset\}$. We write st and lk instead of st_Δ and lk_Δ (for short). We see that $st\, F$ is a subcomplex of Δ, $lk\, F$ a subcomplex of $st\, F$, and that $st\, F = lk\, F = \emptyset$ if $F \notin \Delta$. For $W \subseteq V$, we denote by Δ_W the simplicial complex restricted to W. i.e. the simplicial complex consisting of all faces $F \in \Delta$ whose vertices belong to W.

Now in order to compute $H^i(C_{(a,b)}^\bullet)$, we prove the following

Lemma 1.2. *For all $a \in \mathbb{Z}_+^m$ and $b \in \mathbb{Z}^n$ there exists an isomorphism of complexes*

$$(C^\bullet)_{(a,b)} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}(lk_{st\, H_b} G_b \cup N_a)_W[-j-1]; K), \quad j = |G_b|$$

Proof. The assignment $F \mapsto F' = F - G_b$ establishes a bijection between the set

$$\beta = \{F \in \Delta_W : F \supset G_b, F \cup H_b \cup N_a \in \Delta, |F| = i\}.$$

and the set $\beta' = \{F' \in \Delta_W : F' \in (lk_{st\, H_b} G_b \cup N_a)_W, |F'| = i - j\}$. Here $F' \in (lk_{st\, H_b} G_b \cup N_a)_W$, since $F' \cap (G_b \cup N_a) = \emptyset$ and $F' \cup (G_b \cup N_a) \in st\, H_b$. Therefore we see that

$$\alpha^i : (C^i)_{(a,b)} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}(lk_{st\, H_b} G_b \cup N_a)_{i-j-1}; K), \quad b_F \mapsto \varphi_{F-G_b}$$

is an isomorphism of vector spaces. Here $\varphi_{F'}$ is defined by

$$\varphi_{F'}(F'') = \begin{cases} 1 & \text{if } F = F'', \\ 0 & \text{otherwise.} \end{cases}$$

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$\square$

As a generalization of Hochster's formula [3, Theorem 5.3.8] we prove the following

Theorem 1.3. *Let $I \subset S = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a squarefree monomial ideal with the natural $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading. Then the bigraded Hilbert series of the local cohomology modules of $R = S/I = K[\Delta]$ with respect to the $\mathbb{Z}^m \times \mathbb{Z}^n$-bigrading is given by*

$$H_{H_P^i(K[\Delta])}(\mathbf{s}, \mathbf{t}) = \sum_{F \in \Delta_W} \sum_{G \subset V} \dim_K \widetilde{H}_{i-|F|-1}((\operatorname{lk} F \cup G)_W; K) \prod_{v_i \in G} \frac{s_i}{1 - s_i} \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}}$$

where $\mathbf{s} = (s_1, \ldots, s_m)$, $\mathbf{t} = (t_1, \ldots, t_n)$, $P = (y_1, \ldots, y_n)$ and Δ is the simplicial complex corresponding to the Stanley-Reisner ring $K[\Delta]$.

Proof. By (2), Lemma 1.2 and (3) we observe that there are isomorphisms of bigraded K-vector spaces

$$
\begin{aligned}
H_P^i(K[\Delta])_{(a,b)} &\cong H^i(\operatorname{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}((\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W)[-j-1]; K), \quad j = |G_b| \\
&= H^{i-|G_b|-1}(\operatorname{Hom}_{\mathbb{Z}}(\tilde{\mathcal{C}}((\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W); K) \\
&= \widetilde{H}^{i-|G_b|-1}((\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W; K),
\end{aligned}
$$

and therefore by [3, Exercise 5.3.11] we have

(4) $\qquad \dim_K H_P^i(K[\Delta])_{(a,b)} = \dim_K \widetilde{H}_{i-|G_b|-1}((\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W; K).$

If $H_b \neq \emptyset$ by [3, Lemma 5.3.5], $\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a$ is acyclic, and so $\widetilde{H}_{i-|G_b|-1}((\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a)_W; K) = 0$ for all i. If $H_b = \emptyset$, then $\operatorname{st} H_b = \Delta$, and so $\operatorname{lk}_{\operatorname{st} H_b} G_b \cup N_a = \operatorname{lk} G_b \cup N_a$. Thus in this case $\operatorname{supp}(b) = G_b$. We also note that $H_b = \emptyset$ if and only if $b \in \mathbb{Z}_-^n$. In order to simplify notation we will write $s(a)$ and $s(b)$ for the support of $a \in \mathbb{Z}_+^m$ and $b \in \mathbb{Z}_-^n$, respectively and set $d(i, s(b), s(a)) = \dim_K \widetilde{H}_{i-|s(b)|-1}((\operatorname{lk}_{\operatorname{st} H_b} s(b) \cup$

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$s(a))_W; K)$. Using these facts and (4) we have

$$H_{H^i_P(K[\Delta])}(\mathbf{s}, \mathbf{t}) = \sum_{a\in\mathbb{Z}^m_+, b\in\mathbb{Z}^n_-} \dim_K H^i_P(K[\Delta])_{(a,b)}\mathbf{s}^a\mathbf{t}^b$$

$$= \sum_{b\in\mathbb{Z}^n_-}(\sum_{a\in\mathbb{Z}^m_+} d(i, s(b), s(a))\mathbf{s}^a)\mathbf{t}^b$$

$$= \sum_{b\in\mathbb{Z}^n_-}(\sum_{G\subset V}\sum_{\substack{s(a)=G\\ a\in\mathbb{Z}^m_+}} d(i, s(b), s(a))\mathbf{s}^a)\mathbf{t}^b$$

$$= \sum_{b\in\mathbb{Z}^n_-}(\sum_{G\subset V} d(i, s(b), s(a)) \sum_{s(a)=G}\mathbf{s}^a)\mathbf{t}^b$$

$$= \sum_{b\in\mathbb{Z}^n_-}(\sum_{G\subset V} d(i, s(b), G) \prod_{v_i\in G}\frac{s_i}{1-s_i})\mathbf{t}^b$$

$$= \sum_{\substack{F\in\Delta_W\\ b\in\mathbb{Z}^n_-}}\sum_{s(b)=F}(\sum_{G\subset V} d(i, s(b), G) \prod_{v_i\in G}\frac{s_i}{1-s_i})\mathbf{t}^b$$

$$= \sum_{F\in\Delta_W}\sum_{G\subset V} d(i, F, G) \prod_{v_i\in G}\frac{s_i}{1-s_i} \prod_{w_j\in F}\frac{t_j^{-1}}{1-t_j^{-1}},$$

as desired. Here $\mathbf{s}^a = s_1^{a_1}\ldots s_m^{a_m}$ for $a = (a_1,\ldots,a_m)$ and $\mathbf{t}^b = t_1^{b_1}\ldots t_n^{b_n}$ for $b = (b_1,\ldots,b_n)$. We also used the fact that $\sum_{s(a)=G}\mathbf{s}^a = 1$ for $G = \emptyset$ and $\sum_{s(a)=G}\mathbf{s}^a = \prod_{v_i\in G}\frac{s_i}{1-s_i}$ for $G \neq \emptyset$. $\square$

We observe that Hochster's formula [3, Theorem 5.3.8] easily follows from Theorem 1.3. In fact, if we assume that $m = 0$, then $G = \emptyset$, $(\mathrm{lk}\, F \cup G)_W = \mathrm{lk}\, F$ and $\prod_{v_i\in G} s_i/(1-s_i) = 1$. Moreover, we may consider $\deg Y_j = 1$ for all j. Therefore we get the Hochster formula.

In view of Theorem 1.3 and (4) we get the following isomorphism of K-vector spaces

Corollary 1.4. *For all $a \in \mathbb{Z}^m_+$ and $b \in \mathbb{Z}^n_-$ we have*

$$H^i_P(K[\Delta])_{(a,b)} \cong \widetilde{H}^{i-|F|-1}((\mathrm{lk}\, F \cup G)_W; K),$$

where $F = \mathrm{supp}\, b$ and $G = \mathrm{supp}\, a$.

Corollary 1.5. *With the notation of Theorem 1.3 one has*

$$H_{H^i_P(K[\Delta])}(\mathbf{s}, \mathbf{t}) = H_{H^i_{\mathfrak{m}}(K[\Delta_W])}(\mathbf{t}) + \sum_{F\in\Delta_W}\sum_{\substack{G\subset V\\ G\neq\emptyset}} d(i, F, G) \prod_{v_i\in G}\frac{s_i}{1-s_i} \prod_{w_j\in F}\frac{t_j^{-1}}{1-t_j^{-1}},$$

where $H_{H^i_{\mathfrak{m}}(K[\Delta_W])}(\mathbf{t})$ is the Hilbert series of ith ordinary local cohomology of $K[\Delta_W]$ with respect to the maximal ideal $\mathfrak{m} = (y_1,\ldots,y_n)$ and where

$$d(i, F, G) = \dim_K \widetilde{H}_{i-|F|-1}((\mathrm{lk}\, F \cup G)_W; K).$$

Proof. By Theorem 1.3 we may write

$$H_{H_P^i(K[\Delta])}(\mathbf{s}, \mathbf{t}) = \sum_{F \in \Delta_W} d(i, F, \emptyset) \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}} \quad +$$

$$\sum_{F \in \Delta_W} \sum_{\substack{G \subset V \\ G \neq \emptyset}} d(i, F, G) \prod_{v_i \in G} \frac{s_i}{1 - s_i} \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}}.$$

Since

$$d(i, F, \emptyset) = \dim_K \widetilde{H}_{i-|F|-1}((\operatorname{lk} F \cup \emptyset)_W; K) = \dim_K \widetilde{H}_{i-|F|-1}((\operatorname{lk}_{\Delta_W} F; K),$$

Hochster's formula ([3, Theorem 5.3.8]) implies the desired equality. $\square$

In view of Corollary 1.5 we immediately obtain

Corollary 1.6. $H_P^i(K[\Delta]) \neq 0$ *for* $i = \operatorname{depth} K[\Delta_W]$ *and* $i = \dim K[\Delta_W]$.

We are interested in the Hilbert series of $H_P^i(K[\Delta])$ as a $\mathbb{Z} \times \mathbb{Z}$-bigraded algebra. Note that for all $k, j \in \mathbb{Z}$ we have

$$(5) \qquad H_P^i(K[\Delta])_{(k,j)} = \bigoplus_{\substack{a \in \mathbb{Z}^m, |a|=k \\ b \in \mathbb{Z}^n, |b|=j}} H_P^i(K[\Delta])_{(a,b)},$$

where $|a| = \sum_{i=1}^m a_i$ for $a = (a_1, \ldots, a_m)$ and $|b| = \sum_{i=1}^n b_i$ for $b = (b_1, \ldots, b_n)$.

Using this observation we obtain

Proposition 1.7. *For all* i *and* $k, j \in \mathbb{Z}$ *one has*

$$\dim_K H_P^i(K[\Delta])_{(k,j)} = \sum_{\substack{F \in \Delta_W \\ G \subset V}} d(i, F, G) \binom{k-1}{|G|-1} \binom{-j-1}{|F|-1},$$

where

$$d(i, F, G) = \dim_K \widetilde{H}_{i-|F|-1}((\operatorname{lk} F \cup G)_W; K).$$

Proof. We set $|G| = g$ and $|F| = f$. In view of (5) it follows that the Hilbert series of $H_P^i(K[\Delta])$ with respect to the $\mathbb{Z} \times \mathbb{Z}$-bigraded is obtained from Theorem 1.3 by replacing all s_i and t_j by s and t, respectively. Thus we have

$$H_{H_P^i(K[\Delta])}(s, t) = \sum_{F \in \Delta_W} \sum_{G \subset V} d(i, F, G) \left(\frac{s}{1-s}\right)^g \left(\frac{t^{-1}}{1-t^{-1}}\right)^f.$$

We note that

$$(6) \qquad \frac{1}{(1-s)^i} = \sum_{r=0}^{\infty} \binom{i+r-1}{i-1} s^r \quad \text{for all} \quad i > 0.$$

Expanding $(\frac{s}{1-s})^g$ for $g = 0$ and $g = 1$ and comparing coefficients with (6) we are forced to make the following convention: $\binom{-1}{-1} = 1$, $\binom{i}{-1} = 0$ for all $i \geq 0$ and $\binom{i}{0} = 1$ for all $i \geq 0$. Thus we have

$$H_{H_P^i(K[\Delta])}(s, t) = \sum_{F \in \Delta_W} \sum_{G \subset V} d(i, F, G) \sum_{r=0}^{\infty} \binom{g+r-1}{g-1} s^{r+g} \sum_{h=0}^{\infty} \binom{f+h-1}{f-1} t^{-f-h}.$$

We set $k = r + g$ and $j = -f - h$. Thus $r = k - g$ and $h = -j - f$. Therefore for all k and j with $g \leq k$ and $0 \leq f \leq -j$ the desired formula follows. $\square$

Corollary 1.8. *For all i and $j \in \mathbb{Z}$ one has*

$$\dim_K H_P^i(K[\Delta])_{(0,j)} = \dim_K H_{\mathfrak{m}}^i(K[\Delta_W])_j,$$

where $H_{\mathfrak{m}}^i(K[\Delta_W])$ is the ith ordinary local cohomology of $K[\Delta_W]$ with respect to $\mathfrak{m} = (y_1, \ldots, y_n)$.

Proof. By Proposition 1.7 and the fact that $(\mathrm{lk}\, F)_W = \mathrm{lk}_{\Delta_W} F$ we have

$$
\begin{aligned}
\dim_K H_P^i(K[\Delta])_{(0,j)} &= \sum_{F \in \Delta_W} \dim_K \widetilde{H}_{i-|F|-1}((\mathrm{lk}\, F)_W; K) \binom{-j-1}{|F|-1} \\
&= \sum_{F \in \Delta_W} \dim_K \widetilde{H}_{i-|F|-1}(\mathrm{lk}_{\Delta_W} F; K) \binom{-j-1}{|F|-1} \\
&= \dim_K H_{\mathfrak{m}}^i(K[\Delta_W])_j.
\end{aligned}
$$

The last equality follows from Hochster's theorem. $\square$

For all $j \in \mathbb{Z}$, we set

$$H_P^i(K[\Delta])_j = \bigoplus_k H_P^i(K[\Delta])_{(k,j)},$$

and consider $H_P^i(K[\Delta])_j$ as a finitely generated graded R_0-module. In the following we show that the Krull-dimension of $H_P^i(K[\Delta])_j$ is constant for $j \ll 0$.

Theorem 1.9. *For all i there exists an integer j_0 such that for $j \leq j_0$, the Krull-dimension $\dim H_P^i(K[\Delta])_j$ is constant.*

Proof. By Proposition 1.7 and using (6) the $\mathbb{Z}$-graded Hilbert series of $H_P^i(K[\Delta])_j$ is given by

$$
\begin{aligned}
H_{H_P^i(K[\Delta])_j}(s) \quad &= \quad \sum_{k=0}^{\infty} \dim_K H_P^i(K[\Delta])_{(k,j)} s^k \\
&= \quad \sum_{k=0}^{\infty} \sum_{\substack{F \in \Delta_W \\ G \subset V}} d(i,F,G) \binom{k-1}{|G|-1} \binom{-j-1}{|F|-1} s^k \\
&= \quad \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=0}} d(i,F,G) \binom{-j-1}{|F|-1} \sum_{r=-1}^{\infty} \binom{r}{-1} s^{r+1} \\
&+ \quad \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=1}} d(i,F,G) \binom{-j-1}{|F|-1} \sum_{r=0}^{\infty} \binom{r}{0} s^{r+1} \\
&+ \cdots + \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=m}} d(i,F,G) \binom{-j-1}{|F|-1} \sum_{i=m-1}^{\infty} \binom{r}{m-1} s^{r+1} \\
&= \quad A_0(j) + \frac{A_1(j)s}{1-s} + \frac{A_2(j)s^2}{(1-s)^2} + \cdots + \frac{A_m(j)s^m}{(1-s)^m} \\
&= \quad \frac{\sum_{r=0}^{m} A_r(j)(1-s)^{m-r} s^r}{(1-s)^m},
\end{aligned}
$$

where

$$
A_r(j) = \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=r}} d(i,F,G) \binom{-j-1}{|F|-1}.
$$

We may write

$$
\sum_{r=0}^{m} A_r(j)(1-s)^{m-r} s^r = \sum_{r=0}^{m} B_r(j) s^r,
$$

where $B_r(j)$ is a polynomial with coefficients in $\mathbb{Q}$ of degree at most $n-1$. Therefore we have

(7)
$$
H_{H_P^i(K[\Delta])_j}(s) = \frac{Q_j(s)}{(1-s)^m},
$$

where $Q_j(s) = \sum_{r=0}^{m} B_r(j) s^r$. We denote by $Q_j(s)^{(k)}$ the kth derivative of $Q_j(s)$ as a function in s and set $R_k(j) = [Q_j(s)^{(k)}](1)$ which is of course a polynomial in j. Here we distinguish two cases: First suppose that $R_k = 0$ for all $k \geq 0$. Then the Taylor expansion of $Q_j(s)$

$$
Q_j(s) = \frac{R_0(j)}{0!} + \frac{R_1(j)}{1!}(1-s) + \frac{R_2(j)}{2!}(1-s)^2 + \ldots
$$

implies that $Q_j(s) = 0$ for all j. Thus we see that $R_k = 0$ for all $k \geq 0$, which is equivalent to say that $Q_j(s) = 0$ for all j, and which in turn implies that $H_P^i(K[\Delta])_j = 0$

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for all j, and we set $\dim H_P^i(K[\Delta])_j = -\infty$. Now we assume that not all $R_k = 0$, and define
$$c = \min\{i : R_i \neq 0\}.$$
Thus $R_k(j) = 0$ for all j and all $k < c$. Since R_c has only finitely many zeroes, it follows that $R_c(j) \neq 0$ for $j \ll 0$, i.e. there exists an integer j_0 such that $R_c(j) \neq 0$ for $j \leq j_0$. Therefore $R_k(j) = 0$ for $j \leq j_0$, if $k < c$ and $R_k(j) \neq 0$ for $j \leq j_0$, if $k = c$. Thus for $j \leq j_0$ we may write $Q_j(s) = (1-s)^c \tilde{Q}_j(s)$ where $\tilde{Q}_j(s)$ is a polynomial in s with $\tilde{Q}_j(1) \neq 0$. Therefore by (7) and [3, Corollary 4.1.8] we have $\dim H_P^i(K[\Delta])_j = m - c$ for all $j \leq j_0$, as desired. $\qquad\square$

Definition 1.10. Let R be a positively graded Noetherian ring. A graded R-module N is called *tame*, if there exists an integer j_0 such that either
$$N_j = 0 \quad \text{for all} \quad j \leq j_0, \quad \text{or} \quad N_j \neq 0 \quad \text{for all} \quad j \leq j_0.$$

For example, any finitely generated R-module is tame.

Corollary 1.11. *Let $I \subset S = K[Y_1, \ldots, Y_r]$ be a graded squarefree monomial ideal and let Δ be the simplicial complex such that $K[\Delta] = S/I$. Let P be a monomial prime ideal of $K[\Delta]$. Then for all i, the local cohomology modules of $H_P^i(K[\Delta])$ are tame.*

Proof. The assertion follows from Theorem 1.9. $\qquad\square$

2. Local cohomology of monomial ideals with respect to monomial prime ideals

We recall two results due to Takayama [6]. Let $S = K[Y_1, \ldots, Y_n]$ be a polynomial ring with the standard grading. For a monomial ideal $I \subset S$ we set $R = S/I$. We denote by y_i the image of Y_i in R for $i = 1, \ldots, n$ and set $\mathfrak{m} = (y_1, \ldots, y_n)$, the unique maximal ideal. For a monomial ideal $I \subset S$, we denote by $G(I)$ the minimal set of monomial generators. Let $u = Y_1^{c_1} \cdots Y_n^{c_n}$ be a monomial with $c_j \geq 0$ for all j, then we define $\nu_j(u) = c_j$ for $j = 1, \ldots, n$, and $\operatorname{supp}(u) = \{j : c_j \neq 0\}$.

We set $G_b = \{j : b_j < 0\}$ for $b \in \mathbb{Z}^n$. By Takayama we have

Lemma 2.1. *Let $y = y_{i_1} \cdots y_{i_r}$ with $i_1 < \cdots < i_r$ and set $F = \{i_1, \ldots, i_r\}$. For all $b \in \mathbb{Z}^n$ we have $\dim_K(R_y)_b \leq 1$ and the following are equivalent*

   (i) $(R_y)_b \cong K$
   (ii) $F \supset G_b$ and for all $u \in G(I)$ there exists $j \notin F$ such that $\nu_j(u) > b_j \geq 0$.

For any $b \in \mathbb{Z}^n$, we define a simplicial complex
$$\Delta_b = \left\{ F - G_b \;\middle|\; \begin{array}{l} F \supset G_b, \text{ and for all } u \in G(I) \text{ there exists } j \notin F \\ \text{such that } \nu_j(u) > b_j \geq 0 \end{array} \right\}.$$

Theorem 2.2. *Let $I \subset S = K[Y_1, \ldots, Y_n]$ be a monomial ideal. Then the multi-graded Hilbert series of the local cohomology modules of $R = S/I$ with respect to the $\mathbb{Z}^n$-grading is given by*
$$\operatorname{Hilb}(H_{\mathfrak{m}}^i(R), \mathbf{t}) = \sum_{F \in \Delta} \sum \dim_K \tilde{H}_{i-|F|-1}(\Delta_b; K) \mathbf{t}^b$$

where $\mathbf{t} = (t_1, \cdots, t_n)$. *The second sum runs over* $a \in \mathbb{Z}^n$ *such that* $G_b = F$ *and* $b_j \leq \rho_j - 1$, $j = 1, \ldots, n$, *with* $\rho_j = \max\{\nu_j(u) : u \in G(I)\}$ *for* $j = 1, \ldots, n$, *and* Δ *is the simplicial complex corresponding to the Stanley-Reisner ideal* $\sqrt{I}$.

Note that Takayama's formula can be rewritten as following

$$
\mathrm{Hilb}(H_{\mathfrak{m}}^i(R), \mathbf{t}) = \sum_{F \in \Delta} \dim_K \tilde{H}_{i-|F|-1}(\Delta_b; K) \sum_{\substack{b \in \mathbb{Z}^m, G_b = F \\ b_j \leq \rho_j - 1 \\ j = 1, \ldots, m}} \mathbf{t}^b
$$

$$
= \sum_{F \in \Delta} \dim_K \tilde{H}_{i-|F|-1}(\Delta_b; K) \prod_{w_j \in F} \frac{1 - t_j{}^{\rho_j}}{1 - t_j} \prod_{w_j \notin F} \frac{t_j{}^{-1}}{1 - t_j{}^{-1}}.
$$

We see that in the squarefree case, this formula together with [6, Corollary 1] implies again Hochster's formula.

Now let $S = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a standard bigraded polynomial ring over K. For a monomial ideal $I \subset S$ we set $R = S/I$. The residue classes of the variables will be denoted by x_i and y_j and set $P = (y_1, \ldots, y_n)$. For monomial $u \in S$ we may write $u = u_1 u_2$ where u_1 and u_2 are monomials in X and Y. Let Δ be the simplicial complex corresponding to $\sqrt{I}$. As before we denote the vertices corresponding to the X_i by v_i and those corresponding to the Y_j by w_j. We set $G_b = \{w_j : b_j < 0\}$ for $b \in \mathbb{Z}^n$. With the same arguments as Lemma 2.1 we have

Lemma 2.3. *Let* $y = y_{i_1} \cdots y_{i_r}$ *with* $i_1 < \cdots < i_r$ *and set* $F = \{i_1, \ldots, i_r\}$. *For all* $a \in \mathbb{Z}^m$ *and* $b \in \mathbb{Z}^n$ *we have* $\dim_K(R_y)_{(a,b)} \leq 1$ *and the following are equivalent*

   (i) $(R_y)_{(a,b)} \cong K$
   (ii) $F \supset G_b$, $a \in \mathbb{Z}_+^m$, *and for all* $u \in G(I)$ *there exists* $j \notin F$ *such that* $\nu_j(u_2) > b_j \geq 0$ *or for at least one* i, $\nu_i(u_1) > a_i \geq 0$.

For any $a \in \mathbb{Z}_+^m$ and $b \in \mathbb{Z}^n$, we define a simplicial complex

$$
\Delta_{(a,b)} = \left\{ F - G_b \;\middle|\; \begin{array}{l} F \supset G_b, a \in \mathbb{Z}_+^m \text{ and for all } u \in G(I) \text{ there exists } j \notin F \\ \text{such that } \nu_j(u_2) > b_j \geq 0, \text{or for at least one i}, \nu_i(u_1) > a_i \geq 0 \end{array} \right\}.
$$

As a generalization of Takayama's result we have

Theorem 2.4. *Let* $I \subset S = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ *be a monomial ideal with the natural* $\mathbb{Z}^m \times \mathbb{Z}^n$-*bigrading. Then the bigraded Hilbert series of the local cohomology modules of* $R = S/I$ *with respect to the* $\mathbb{Z}^m \times \mathbb{Z}^n$-*bigrading is given by*

$$
\mathrm{Hilb}(H_P^i(R), \mathbf{s}, \mathbf{t}) =
$$

$$
\sum_{F \in \Delta_W} \sum_{G \subset V} D(i, F, G) \prod_{v_i \notin G} \frac{1 - s_i{}^{-\sigma_i}}{1 - (s_i)^{-1}} \prod_{v_i \in G} \frac{s_i}{1 - s_i} \prod_{w_j \notin F} \frac{1 - t_j{}^{\rho_j}}{1 - t_j} \prod_{w_j \in F} \frac{t_j{}^{-1}}{1 - t_j{}^{-1}},
$$

where $P = (y_1, \ldots, y_n)$, $D(i, F, G) = \dim_K \tilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K)$, $\mathbf{s} = (s_1, \cdots, s_m)$, $\mathbf{t} = (t_1, \cdots, t_n)$, $\rho_j = \max\{\nu_j(u_2) : u \in G(I)\}$ *for* $j = 1, \ldots, n$, $\sigma_i = \max\{\nu_i(u_1) : u \in G(I)\}$ *for* $j = 1, \ldots, m$, *and* Δ *is the simplicial complex corresponding to the Stanley-Reisner ideal* $\sqrt{I}$.

Proof. With the same arguments as in the proof Theorem 1 in [6] we can show that

$$\mathrm{Hilb}(H_P^i(R), \mathbf{s}, \mathbf{t}) = \sum \sum \dim_K \tilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K) \mathbf{s}^a \mathbf{t}^b$$

where the first sum runs over $F \in \Delta_W$, $b \in \mathbb{Z}^n$ such that $G_b = F$ and $b_j \leq \rho_j - 1$, $j = 1, \ldots, n$, and the second sum runs over $a \in \mathbb{Z}^m$ such that $N_a = G$ and $a_i \geq \sigma_i - 1$, $i = 1, \ldots, m$. Indeed, if we assume that $b_j > \rho_j - 1$ and $a_i < \sigma_i - 1$, the proof of [6, Theorem 1] shows that $\dim_K \tilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K) = 0$ for all i. Therefore we may write

$$\mathrm{Hilb}(H_P^i(R), \mathbf{s}, \mathbf{t}) = \sum_{\substack{F \in \Delta_W}} \sum_{\substack{G \subset V}} D(i, F, G) \sum_{\substack{a \in \mathbb{Z}^m, N_a = G \\ a_i \geq \sigma_i - 1 \\ i=1,\ldots,m}} \mathbf{s}^a \sum_{\substack{b \in \mathbb{Z}^n, G_b = F \\ b_j \leq \rho_j - 1 \\ j=1,\ldots,n}} \mathbf{t}^b,$$

where

$$D(i, F, G) = \dim_K \tilde{H}_{i-|F|-1}(\Delta_{(a,b)}; K).$$

Since

$$\sum_{\substack{a \in \mathbb{Z}^m, N_a = G \\ a_i \geq \sigma_i - 1 \\ i=1,\ldots,m}} \mathbf{s}^a = \prod_{v_i \notin G} \frac{1 - s_i^{-\sigma_i}}{1 - (s_i)^{-1}} \prod_{v_i \in G} \frac{s_i}{1 - s_i}$$

and

$$\sum_{\substack{b \in \mathbb{Z}^n, G_b = F \\ b_j \leq \rho_j - 1 \\ j=1,\ldots,n}} \mathbf{t}^b = \prod_{w_j \notin F} \frac{1 - t_j^{\rho_j}}{1 - t_j} \prod_{w_j \in F} \frac{t_j^{-1}}{1 - t_j^{-1}},$$

the desired formula follows. $\square$

We observe that Theorem 1.3 is a special case of Theorem 2.4. In fact, if we assume that $\sigma_i = 1$ for $i = 1, \ldots, m$ and $\rho_j = 1$ for $j = 1, \ldots, n$, then $a \in \mathbb{Z}_+^m$, $b \in \mathbb{Z}_-^n$, $\prod_{v_i \notin G} \frac{1 - s_i^{-\sigma_i}}{1 - (s_i)^{-1}} = 1$ and $\prod_{w_j \notin F} \frac{1 - t_j^{\rho_j}}{1 - t_j} = 1$. Moreover, by the proof of [6, Corollary 1] we have $\Delta_{(a,b)} = \mathrm{lk}_{\mathrm{st}\, H_{(a,b)}} G_{(a,b)} = \mathrm{lk}_{\mathrm{st}\, N_a \cup H_b} G_b = \mathrm{lk}_{\mathrm{st}\, N_a} G_b = (\mathrm{lk}\, F \cup G)_W$.

Proposition 2.5. *For all i and $k, j \in \mathbb{Z}$ one has*

$$\dim_K H_P^i(R)_{(k,j)} = \sum_{\substack{F \in \Delta_W \\ G \subset V}} D(i, F, G) \sum_{r=0}^{\sigma_G} a_G(r) \binom{k+r-1}{|G|-1} \sum_{h=0}^{\rho_F} b_F(h) \binom{h-j-1}{|F|-1},$$

where $\sigma_G = \sum_{v_i \notin G}(\sigma_i - 1)$, $\rho_F = \sum_{w_j \notin F}(\rho_j - 1)$ and $a_G(r), b_F(h) \in \mathbb{Z}$ for $r = 0, \ldots, \sigma_G$ and $h = 0, \ldots, \rho_F$.

Proof. In Theorem 2.4 we replace all s_i by s and all t_j by t, and obtain

$$H_{H_P^i(R)}(s, t) = \sum_{\substack{F \in \Delta_W}} \sum_{\substack{G \subset V}} D(i, F, G) P_G(s^{-1}) \left(\frac{s}{1-s}\right)^{|G|} Q_F(t) \left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|},$$

where

$$P_G(s^{-1}) = \prod_{v_i \notin G} (1 + s^{-1} + \cdots + s^{-\sigma_i+1}) \quad and \quad Q_F(t) = \prod_{w_j \notin F} (1 + t^1 + \cdots + t^{\rho_j-1})$$

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with $\deg P_G(s^{-1}) = \sigma_G$ and $\deg Q_F(t) = \rho_F$. We may write $P_G(s^{-1}) = \sum_{r=0}^{\sigma_G} a_G(r)s^{-r}$ where $a_G(r) \in \mathbb{Z}$ for $r = 0, \ldots, \sigma_G$ and $Q_F(t) = \sum_{h=0}^{\rho_F} b_F(h)t^h$ where $b_F(h) \in \mathbb{Z}$ for $h = 0, \ldots, \rho_F$. By setting $|G| = g$ and $|F| = f$, we have

$$H_{H_P^i(R)}(s,t) = \sum_{F \in \Delta_W} \sum_{G \subset V} D(i, F, G) A_G(s) B_F(t),$$

where

$$A_G(s) = \sum_{r=0}^{\sigma_G} a_G(r) \sum_{l=0}^{\infty} \binom{g+l-1}{g-1} s^{l+g-r}$$

and

$$B_F(t) = \sum_{h=0}^{\rho_F} b_F(h) \sum_{\rho=0}^{\infty} \binom{f+\rho-1}{f-1} t^{-f-\rho+h}.$$

We set $k = l + g - r$ and $j = -f - \rho + h$. Then $r = l + g - k$ and $h = j + f + \rho$, and the desired formula follows. $\qquad\square$

Theorem 2.6. *For all i there exists an integer j_0 such that for $j \leq j_0$, the Krull-dimension $\dim H_P^i(R)_j$ is constant.*

Proof. By Proposition 2.5 the $\mathbb{Z}$-graded Hilbert series of $H_P^i(R)_j$ is given by

$$
\begin{aligned}
H_{H_P^i(R)_j}(s) &= \sum_{k=0}^{\infty} \dim_K H_P^i(R)_{(k,j)} s^k \\
&= \sum_{k=0}^{\infty} \sum_{\substack{F \in \Delta_W \\ G \subset V}} D(i, F, G) \sum_{t=0}^{\sigma_G} a_G(t) \binom{k+t-1}{|G|-1} \sum_{h=0}^{\rho_F} b_F(h) \binom{h-j-1}{|F|-1} s^k \\
&= A_0(j) \sum_{r=-1}^{\infty} \sum_{t=0}^{\sigma_G} a_G(t) \binom{t+r}{-1} s^{r+1} \\
&\quad + A_1(j) \sum_{r=-1}^{\infty} \sum_{t=0}^{\sigma_G} a_G(t) \binom{t+r}{0} s^{r+1} \\
&\quad + \cdots + A_m(j) \sum_{r=-1}^{\infty} \sum_{t=0}^{\sigma_G} a_G(t) \binom{t+r}{m-1} s^{r+1} \\
&= A_0(j) + A_1(j)\frac{P_1(s)}{1-s} + A_2(j)\frac{P_2(s)}{(1-s)^2} + \cdots + A_m(j)\frac{P_m(s)}{(1-s)^m} \\
&= \frac{\sum_{r=0}^{m} A_r(j)(1-s)^{m-r} P_r(s)}{(1-s)^m},
\end{aligned}
$$

where

$$A_r(j) = \sum_{\substack{F \in \Delta_W \\ G \subset V, |G|=r}} D(i, F, G) \sum_{h=0}^{\rho_F} b_F(h) \binom{h-j-1}{|F|-1},$$

and $P_r(s) \in \mathbb{Z}[s]$ with $\deg P_r(s) = r$. Here we used (6) and that $a_G(0) = 1$ for $G = \emptyset$, $\binom{t+r}{-1} = 1$ for $t + r = -1$ and 0 otherwise, $\binom{t+r}{0} = 1$ for $t + r \geq 0$ and

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$\binom{t+r}{n} = \binom{t}{n} + r\binom{t}{n-1} + \frac{r(r-1)}{2!}\binom{t}{n-2} + \frac{r(r-1)(r-2)}{3!}\binom{t}{n-3} + \cdots + \frac{r(r-1)(r-2)\ldots(r-n+1)}{n!}\binom{t}{0}$ for $t+r \geq n \geq 1$. We may write

$$\sum_{r=0}^{m} A_r(j)(1-s)^{m-r}P_r(s) = \sum_{r=0}^{m} B_r(j)s^r,$$

where $B_r(j)$ is a polynomial with coefficients in $\mathbb{Q}$ of degree at most $n-1$. Therefore we have

$$H_{H_P^i(R)_j}(s) = \frac{Q_j(s)}{(1-s)^m},$$

where $Q_j(s) = \sum_{r=0}^{m} B_r(j)s^r$. We proceed in the same way as in the proof of Theorem 1.9 and get the desired result. $\qquad\square$

Corollary 2.7. *Let $I \subset S = K[Y_1, \ldots, Y_r]$ be any monomial ideal and set $R = S/I$. Let P be a monomial prime ideal in R. Then for all i, the local cohomology modules of $H_P^i(R)$ are tame.*

Proof. The assertion follows from Theorem 2.6. $\qquad\square$

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